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Actual and Ideal Infinitesimals in Leibniz's *Specimen Dynamicum*

Tzuchien Tho

Institute for Research in the Humanities,
University of Bucharest/Università degli Studi di Milano¹
tzuchien.tho@gmail.com

Abstract: This article aims to treat the question of the reality of Leibniz's infinitesimals from the perspective of their application in his account of corporeal motion. Rather than beginning with logical foundations or mathematical methodology, I analyze Leibniz's use of an allegedly "instantiated" infinitesimal magnitude in his treatment of dead force in the *Specimen Dynamicum*. In this analysis I critique the interpretive strategy that uses the Leibnizian distinction, drawn from the often cited 1706 letter to De Volder, between actual and ideal for understanding the meaning of Leibniz's infinitesimal fictionalism. In particular, I demonstrate the ambiguity that results from sticking too closely with the idea that ideal mathematical terms merely "represent" concrete or actual things. In turn I suggest that, rather than something that had to be prudentially separated from the realm of actual things, the mathematics of infinitesimals was part of how Leibniz conceived of the distinction between the actual and ideal within the *Specimen Dynamicum*.

Keywords: infinitesimal, fictional quantity, syncategorematic, force, continuity, measurement, actual, ideal.

"Between you and me, I think Fontenelle [...] was joking when he said he would derive metaphysical elements from our calculus. To tell the truth, I myself am far from convinced that our infinites and infinitesimals should be considered as anything other than ideals, or well-founded fictions."²

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² Abbreviations to Leibniz editions follow convention:

GM= *Leibnizens Mathematische Schriften*, ed. by C.I. Gerhardt, 7 vols., Berlin and Halle: Asher and Schmidt, 1849-63; reprint ed. Hildesheim: Olms, 1971. Cited by volume and page.

1. Introduction

The controversies that surrounded Leibniz's infinitesimals in his own time echo throughout the history of infinitesimals. Although Leibniz's syncategorematical³ interpretation of infinitesimals was made rigorous and standardized by Cauchy's and Weierstrass' work on functions and, later, by Bolzano's ε - δ definition of the limit, similar sorts of ontological and foundational controversies concerning the status of infinitesimals continued throughout the 20th century. These debates go well into the period of the *Grundlagenkrise der Mathematik* and eventually Robinson's pathbreaking *Non-standard Analysis*. Although this is not the place to treat these controversies, we cannot underestimate how this history of the infinitesimals affects our interpretation of Leibniz's evolution concerning this problem.

Partly due to the fact that we receive Leibniz through this history of controversy, contemporary interpreters rush to vindicate the rigor of Leibniz's concept and usage of infinitesimal terms across his metaphysics, mathematics and mathematical physics. Through these demonstrations of rigor, the vindication of Leibniz's early flirtation with infinitesimal indivisible magnitudes from his pre-1672 writings to his maturation in his *séjour parisien* where a syncategorematic and "fictional" infinitesimal emerged, much of what was uncertain about Leibniz's changing views has been rendered clear. These commentaries along with the editing of Series VII of the *Akademie* edition since 1990 has made Leibniz's confrontation with the labyrinth of the continuum a tractable issue for a new generation of Leibniz interpreters.

Yet what remains limited in the drive to vindicate Leibniz from charges of unrigorousness or contradiction concerning infinitesimals is that this defense often results in a heavily reductive treatment of syncategorematical

GP= *Die philosophischen Schriften von Gottfried Wilhelm Leibniz*, ed. by C.I. Gerhardt, 7 vols. Berlin and Halle: Asher and Schmidt, 1875-90; reprint ed. Hildesheim: Olms, 1965. Cited by volume and page.

A= *Gottfried Wilhelm Leibniz: Sämtliche Schriften und Briefe*, ed. by Deutsche Akademie der Wissenschaften, Darmstadt and Berlin: Akademie Verlag, 1923-. Cited by series, volume and document number.

Leibniz, Letter to Varignon, 20 June 1702, GM IV, p. 110.

³ For the sake of precision, I will refer to Richard T.W. Arthur's description of Leibniz's syncategorematical infinite and infinitesimal. "To assert an infinity of parts syncategorematical-ly is to say that for any finite number x that you choose to number the parts, there is a number of parts y greater than this: $(\forall x)(\exists y)Fx \rightarrow y > x$, with $Fx = x$ is finite, and x and y numbers." Richard T.W. Arthur, "A complete denial of the continuous? Leibniz's law of continuity", *Synthese*, forthcoming, online, <<http://www.humanities.mcmaster.ca/~rarthur/articles.arthur.htm>>, pp. 1-36, p. 7. Arthur notes that it was Hidé Ishiguro's work especially in the 1990 second edition of *Leibniz's Philosophy of Logic and Language* that brought many commentators back to a focus on the syncategorematical infinite. Cf. Hidé Ishiguro, *Leibniz's Philosophy of Logic and Language*, 2nd ed., Cambridge: Cambridge University Press, 1990.

infinitesimal as basically finite. In a number of significant places in Leibniz's maturity (post 1672), we do see how Leibniz reduces his syncategorematic infinitesimal to indefinite finite terms through "infinite" series and algorithms.⁴ There is no doubt that this is an important *aspect* of Leibniz's treatment of infinitesimal terms.

There are then not only one but two (and perhaps more) important episodes in Leibniz's complex relation with infinitesimals. A strict focus on the first episode, his move towards the rejection of actual infinitesimals through his mathematical maturation in Paris, tells only one side of the story. The development of his post-1676 treatment of motion, one that requires the resources of his infinitesimal calculus, is another. As I will examine in further detail below, what Leibniz constructs is this further episode, starting from his January 1678 *De Corporum Concursu*, is a reform of mechanics, explicitly critical of his mechanistic predecessors, that will mature into what Leibniz in his 1695 *Specimen Dynamicum* will call a "New Science of Dynamics" based on force.⁵ Now, although the dynamics projects spans almost two decades of Leibniz's maturity (circa late 1670s to late 1690s) with lengthy treatises circulated in the republic of letters, the sole focus of my investigation here will be on the *Specimen Dynamicum* which stands as a landmark of this lengthy intellectual engagement. The central aim here is to critique the standard understanding of this crucial document of Leibniz's foundational account of physical motion and its relation to the mathematics of the calculus.

Under the shadow of a reductionist reading of infinitesimals, we might be surprised to find the use of infinitesimals applied to actual things. If infinitesimals are ideal and fictional, how is it that they can be used to treat actual things like *force*? Much of what is difficult in resolving this question comes from our habitual separation between what is actual and what Leibniz understands as ideal and fictional. Since interpreters have been habituated in using the notion of "fiction" as a means to distance Leibniz's infinitesimals from actual things or entities, what we have not fully accounted for is how these fictional or ideal terms are then brought back to account for actual things themselves. Here Daniel Garber's recent attempt to interpret the bridging of fictional and ideal infinitesimals with actual things through the notion of "representation" provides an opportunity to rethink this separation.

⁴ A key instance of Leibniz's treatment of (fictional) syncategorematic infinitesimals with infinite series is the quadrature demonstration and the connected treatment of π through what we now call Leibniz (infinite) series in his 1675 *De Quadratura Arithmetica*. Cf. Leibniz, *De Quadratura Arithmetica Circuli Ellipseos et Hyperbolae Cujus Trigonometria Sine Tabulis*, ed. and trans. by Marc Parmentier, Latin text ed. by Eberhard Knobloch, Paris: Librairie J. Vrin, 2004.

⁵ Leibniz, "*Specimen Dynamicum*", GM IV, 234-253; all following citations of "*Specimen Dynamicum*" will be taken from G.W. Leibniz, *Philosophical Essays*, trans. and ed. by Roger Ariew and Daniel Garber, Indianapolis: Hackett Publishing Company, 1989, pp. 117-138, p. 118.

The use of representation to secure the relation between ideal infinitesimals and actual things is motivated by the fear of plunging Leibniz back into the dangers of contradiction engendered by actual infinitesimals. From this position we might say that infinitesimals can represent actual things but are not themselves actual. The fear behind such a prudential reading of Leibniz's infinitesimal in *Specimen Dynamicum* is overstated. The unfortunate framing of the problem of infinitesimals in Leibniz's account of corporeal motion ultimately leads us to downplay the crucial role infinitesimals played in the *Specimen Dynamicum* and how this role can further shed light on the richness of Leibniz's notion of infinitesimal "fictions".

The move toward seeing how infinitesimals *operate* in Leibniz's thought, more than how they are defined, allows us to interpret Leibniz's *fictional* infinitesimal beyond merely something which is "not actual". In what follows, I will begin first by providing an interpretation of how Leibniz engages a mathematical distinction between the discrete and the continuous to understand a key aspect of the difference between the actual and the ideal. I will then examine an "instantiation" of a seemingly infinitesimal magnitude in the actual in the opening sections of the *Specimen Dynamicum*. I will argue that Garber's reading of these passages, one that interprets ideal things as representing actual things, although it avoids the danger of something like an *actual* infinitesimal, does not in fact allow us to understand much about the use of infinitesimal terms and more generally about the role of the infinitesimal calculus in this text. Through the understanding gained from this critique, I will finally highlight Leibniz's employment of infinitesimal terms and the methods of the calculus to construct an analysis of actual and ideal aspects of his account of corporeal motion itself. I then conclude that rather than a neat separation between actual, physical things that are to be represented and ideal, mathematical terms that do the representing, Leibniz employs mathematics to create a theoretical framework capable of distinguishing between actual and ideal levels within his account of corporeal motion.

2. Actual:Ideal::Discrete:Continuous?

Recent availability of mathematical manuscripts surrounding the infinitesimal calculus has brought new clarity to underlining the methodological rigor of Leibniz's reasoning with infinitesimal fictions.⁶ In addition, recent commentaries by Hidé Ishiguro and Richard Arthur have also aimed to argue for the legitimacy of Leibniz's use of these *syncategorematic* infinite and infinitesimals in

⁶ Cf. Marc Parmentier, "Introduction," to Leibniz, *Quadrature arithmétique du cercle, de l'ellipse et de l'hyperbole et la trigonométrie sans table trigonométriques qui en est le corollaire*, pp. 7-32. Cf. Eberhard Knobloch, "Leibniz's Rigorous Foundation of Infinitesimal Geometry by Means of Riemannian Sums," *Synthese* 133. 3 (2002), pp. 59-73.

question.⁷ What is not obvious in these different attempts at clarifying Leibniz's infinitesimals is how its status as "ideal", as we see stated in his 20 June 1702 letter to Varignon quoted in the epigraph, can establish a relation with actual things. For example, Ishiguro attempts to address Leibniz's use of infinitesimal by according a Fregean standard of "contextual reference" in order to make Leibniz's conception of such mathematical objects as the infinitesimal rigorous.⁸ Ishiguro's post-Fregean assessment of Leibniz's use of infinitesimals is however flawed in one important respect. If we reduce the use of infinitesimal terms in a Fregean manner to, say, natural numbers or other mathematical terms, we might be in a position to justify the rigor of these problematic terms but at the cost of dissolving their peculiar nature into the status of mathematical objects as such. Simply put, if all mathematical terms are ideal then it seems entirely superfluous to add that these well-founded fictions are "ideals".

This new focus on Leibniz's syncategorematical infinitesimals is however a focus on the use of "fiction" insofar as it concerns the *admissibility* of infinitesimal terms into mathematics (calculation and demonstration) and not its wider problems in metaphysics and natural science. It is in the context of this problem of admissibility that the passage cited in the epigraph should be read. Although the initial developments of Leibniz's infinitesimal calculus were already developed during the *séjour parisien*, this project only took an explicit public form much later in the 1684 "*Nova Methodus pro maximis et minimis, itemque tangentibus, et singulare pro illis calculi genus*" and the 1686 "*De Geometria Recondita et analysi indivisibilium atque infinitorum*". In the early 1700's, Leibniz was asked to clarify a heated debate that was going on in Paris around the French Royal Academy of Sciences sparked by the first textbook on the calculus, G. de L'Hôpital's 1696 "*Analyse des infiniment petits*". In this context, Leibniz often played a very conservative role, prudentially maneuvering between defending the method of the infinitesimal calculus that had come under significant attack and playing down its more controversial implications. In the case of this letter to Varignon, the prudential use of "fiction" was rhetorically directed toward the ways in which he felt his theory was over-extended, in this case by Fontenelle, into domains where it did not belong. As such we can see how Leibniz's prudent separation between infinitesimal fictions and actual things in the context of these debates was aimed at treating the admissibility of infinitesimal terms in *mathematics* and to insist that,

[I]t is unnecessary to make mathematical analysis depend on metaphysical controversies or to make sure that there are lines in nature that are infinitely

⁷ Cf. Hidé Ishiguro, *Leibniz's Philosophy of Logic and Language*. Cf. Richard T.W. Arthur, "Actual Infinitesimals in Leibniz's Early Thought," in *The Philosophy of the Young Leibniz*, ed. by Mark Kulstad, Mogens Laerke and David Snyder, Stuttgart: Franz Steiner Verlag, 2009, pp. 11-28.

⁸ Hidé Ishiguro, *Leibniz's Philosophy of Logic and Language*, p. 100.

small in a rigorous sense in contrast to our ordinary lines, or as a result that there are lines infinitely greater than our ordinary ones.⁹

Understanding the context of Leibniz's prudential remarks should temper the idea that infinitesimal fictions were qualified as fictional because of their complete separation from the actual. In a later letter to De Volder on 19 January 1706, Leibniz provides a picture of how the ideal and the actual are to be understood through the lens of the infinite and infinitesimal. He says,

The continuum of course, contains indeterminate parts. But in actual things nothing is indefinite, indeed, every division that can be made has been made in them. Actual things are composed as a number is composed of unities, but ideal things are composed of fractions: there are actually parts in a real whole, but not in an ideal whole. As long as we seek actual parts in the order of possibles and indeterminate parts in aggregates of actual things, we confuse ideal things with real substances and entangle ourselves in the labyrinth of the continuum and inexplicable contradictions.¹⁰

Leibniz warns us that the reason why fictional infinitesimals are ideal is because they belong to the realm of the continuous where they commune with interminable parts of division. Rendering an indefinite division into a unity such as a sum of all the divisions or the smallest part that results from division is precisely, as Leibniz says, to create "inexplicable contradictions". We should notice, however, that this separation between the actual and the ideal through the discrete and continuous is itself a mathematical distinction. In the case of the actual, we find discrete unities and in the case of the ideal, we find continuities that are given to indefinite sub-divisions.

This correspondence with De Volder is usually taken as evidence for Leibniz's separation of the discrete and the continuous by correlating the discrete with the actual and the continuous with the ideal. The warning against confounding the two levels certainly seems to go in this direction. If we look closer however, we find something else at work. In this text, we see that the ideal is not just continuity and the actual is not just discreteness. If we take the realm of mathematical objects to be in the ideal then both the continuous and the discrete exist in the ideal although actual things are true unities. That is, this distinction between the ideal and the actual makes use of an analogy *within* a mathematical framework capable of distinguishing between the discrete and continuous.

When Leibniz highlights the dangers of confounding the ideal and the actual, we understand that he makes this point at the same time as *relating* them

⁹ Leibniz, Letter to Varignon, 2 February 1702, GM IV, p. 91.

¹⁰ Leibniz, Letter to De Volder, 19 January 1706, GP II, p. 282; Leibniz, *Philosophical Essays*, p. 185.

analogically. Hence just when it seems that this warning against confounding the actual and ideal levels appears as a definitive separation, Leibniz explicitly relates them back together. This occurs just a few lines down in this same letter to De Volder,

However, the science of continua, that is, the science of possible things, contains eternal truths, truths which are never violated by actual phenomena, since the difference [between the real and ideal] is always less than any given amount that can be specified. And we don't have, nor should we hope for, any mark of reality in phenomena, but the fact that they agree with one another and with eternal truths.¹¹

Here Leibniz speaks of a converging and negligible difference between the ideal and the actual. We can compare this suggestion of a convergence to his statements on infinitesimals conveyed a few years before to Pinson on 29 August 1701 in a more directly mathematical context, "[I]n lieu of the infinite or infinitely small, we take quantities as great or as small as is required so that the error would be less than the given error [l'erreur soit moindre que l'erreur donnée][...] such that we do not differ from the style of Archimedes except in the expressions which are more direct in our method [...]"¹² What can we make of this convergence while maintaining the lack of "any mark of reality in phenomena"?

In the convergence between the ideal and the actual, Leibniz deepens the analogy used to distinguish the ideal and the actual. In the letter to Pinson, Leibniz rehearses his recurring strategy for justifying his use of infinitesimals; a strategy found in his correspondences with Varignon and Des Bosses, just to name a few often cited instances.¹³ This defense consists in saying that that these terms are not non-Archimedean.¹⁴ These fictional terms differ from the Archimedean precedent only in that these alternative "expressions", as Leibniz puts it, can ultimately be thought of as standard Archimedean terms in calculation with a negligible difference. That is, they differ "less than the given error [moindre que l'erreur donnée]".

Can the convergence between the ideal and the actual really be understood through an analogy of the convergence between infinitesimal and finite terms? It seems that a direct association of these two sets of terms will lead to confusion. After all if all mathematical terms are ideal then fictional infinitesimals

¹¹ Ibid., GP II, p. 282; *Philosophical Essays*, p. 186.

¹² Leibniz, Letter to Pinson, 29 August 1701, A I, 20, N. 290 [author's translation].

¹³ Cf. Leibniz, Letter to Varignon, 20 June 1702, GM IV, p. 106. Letter to Des Bosses, 3 March 1706, GP II, p. 305.

¹⁴ The definition of an Archimedean expression in quantity or magnitude can be defined in its Euclidian manner as quantities that "have a ratio to one another which are capable, when multiplied, of exceeding one another." Euclid, *Elements*, trans. by Thomas L. Heath and Dana Densmore, Santa Fe: Green Lion Press, 2002, p. 99.

would appear to be “doubly” ideal. Here we would lose track of what is supposed to be actual and ideal or what this analogy actually hangs on. I propose that only by looking at a clear and concrete case of a distinction between the ideal and the actual through the *Specimen Dynamicum* could we gain any grasp on this slippery analogy.

Before moving on we should notice a possible terminological confusion. In treating the actual and the ideal, we might be tempted to invoke Leibniz’s larger modal theory. In dealing with instantiated “fictional” terms like the infinite and infinitesimal, we confront a sense of the actual, as we shall examine, that pertains to causes which are more determinate and hence more real than their effects which are the phenomenal aspects of motion. There is of course something modal about this relation between cause and effect but this is not the primary way in which Leibniz is using the terms here. We should be careful then in not conflating this distinction between the actual and ideal with the modal distinction between the actual and the possible (and compossibility). As such our task is to see how Leibniz understands these actual and ideal levels of reality as different but also related. By looking at an instance of the “actual” instantiation of the infinitesimal, we will provide the resources for understanding how the difference between the actual and the ideal converges in a way that renders the infinitesimal calculus central in this account of the actual.

3. *An Actual Instance of the Infinitesimal?*

In our quick look at Leibniz’s 1706 letter to De Volder above, Leibniz first employed a mathematical framework in order to use an intra-mathematical distinction between continuity and discreteness to explain the distinction between ideal and actual things. He then used this analogy to relate the ideal and the actual through the relation between fictional infinitesimals (continuous) and actual things (discrete). To elucidate Leibniz’s analogy that we have been considering, we must turn to the application of the infinitesimals in the domain of Leibniz’s mature mathematical physics. As such, we examine one of the important moments in his mature attempts to account for corporeal motion, the 1695 *Specimen Dynamicum*.

Leibniz’s work on the reform of mechanics would eventually mature into what he calls “*dynamics*”. This is not the place to trace the details of the *dynamics* that occupied Leibniz for roughly two decades between the 1678 *De Corporum Concursu* until his *Essay de Dynamique* around 1700. Instead I will focus on one significant aspect of this project, the *Specimen Dynamicum*, a summary for this new science of force that Leibniz published in the *Acta Eruditorum* in 1695.¹⁵ We will examine some of the background of the *Specimen*

¹⁵ Hereafter *Specimen Dynamicum* will be abbreviated as *Specimen*.

later in this article. Our priority now is rather to see how the infinitesimal is related to actual things.

In the *Specimen*, the crucial moment for our consideration is his treatment of dead force. We begin with this use of an infinitesimal term for treating an actual magnitude. Here Leibniz argues that,

Consider tube AC rotating around the immobile center C on the horizontal plane of this page with a certain uniform speed, and consider ball B in the interior of the tube, just freed from a rope or some other hindrance, and beginning to move by virtue of centrifugal force. It is obvious that, in the beginning, the *conatus* for receding from the center, namely, that by virtue of which the ball B in the tube tends toward the end of tube, A, is infinitely small in comparison with the impetus which it already has from rotation, that is, it is infinitely small in comparison with the impetus by virtue of which the ball B, together with the tube itself, tends to go from place D to (D), while maintaining the same distance from the center. But if the centrifugal impression deriving from the rotation were continued for some time, then by virtue of that very circumstance, a certain complete centrifugal impetus (D)(B), comparable to the rotational impetus D (D), must arise in the ball. From this it is obvious that the *nisus* is two-fold, that is, elementary and infinitely small, which I also call solicitation, and that which is formed from the continuous or repetition of elementary *nisus*, that is, impetus itself.¹⁶

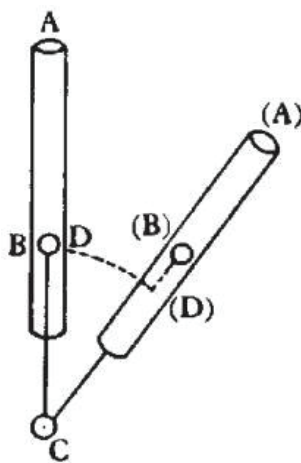


Figure 1¹⁷

This example of centrifugal force presents us with a model for understanding how the infinitesimal is related to force and how force is related to motion.

¹⁶ Leibniz, "Specimen Dynamicum," in *Philosophical Essays*, p. 121.

¹⁷ This figure taken from Leibniz, "Specimen," p. 121.

Let us first understand what Leibniz aimed at presenting with this example. In the example, the rotation of the tube results in a change in place which is the movement of the ball from position or place D to (B). This result is the combination of two distinct forces at work. The motion caused by mere rotation can be isolated from the motion caused by centrifugal force by considering the ball tied by a string to the center C. When there is no string, these two forces work together and move the body away from the center C. As such, the sum of motion can be effectively divided into two distinct forces, the force of rotation and centrifugal force. The two together, centrifugal impetus and rotational impetus, constitute the actual motion of the ball away from the center. In turn, if we isolate centrifugal force from rotational motion, we see that centrifugal force acts in the body as an infinitesimal solicitation to motion without actual motion from the starting position (D) in the tube outward to the end of the tube. As Leibniz explains, this centrifugal striving toward motion is infinitesimal relative to the impetus or quantity of rotational motion. The solicitation or *conatus* does not actually move the ball away from this point of reference, the center C in the tube. To explain this distinction between a force that governs the centrifugal solicitation to move and the force that causes the actual rotational movement of the ball, Leibniz argues respectively that:

One force is elementary, which I call dead force, since motion [*motus*] does not yet exist in it, but only a solicitation to motion [*motus*] as with a ball in the tube, or a stone in a sling while it is still held in by the rope. The other force is ordinary force, joined with actual motion, which I call living force.¹⁸

Dead force then, such as centrifugal force or the gravitational force acting on a suspended body, is a *conatus* that does not necessarily produce motion, even if it gathers the solicitation for motion in the many moments wherein that force is compounded. Living force, on the other hand, is force understood through the compounding of momentaneous *motio* defined as the product of mass and velocity.¹⁹ In turn, Leibniz explained that just as in centrifugal force, the case of the force of a suspended body acted on by gravitation, or heaviness, is one of dead force:

[W]hen we are dealing with impact, which arises from a heavy body which has already been falling for some time [...] the force in question is living force, which arises from an infinity of continual impressions of dead force. And this is what Galileo meant when he said, speaking enigmatically, that the force of impact is infinite in comparison with the simple *nisus* of heaviness.²⁰

¹⁸ Leibniz, "*Specimen Dynamicum*," p. 121.

¹⁹ *Ibid.*, p. 120.

²⁰ *Ibid.*, p. 122.

Leibniz's explanation of what remained enigmatic in Galileo was thus that: The comparison of the compounding of force in a falling body from the solicitation of gravitational acceleration to the force of body manifested as weight (the *nisus* of heaviness) is like the comparison of an infinite to a finite. It is not that the living force of the falling body is infinite. Instead, we see that living force is of a different *order* to dead force in terms of an analogy with this distinction of order between the finite and infinite.²¹ In terms of this distinction between a living force that is actually in motion and a dead force that is merely solicited to move without actually moving, the comparison of the two are accounted through this use of infinite and infinitesimal terms. Here Leibniz provided the occasion for the use of the methods of the infinitesimal calculus in the treatment of actual things. As Leibniz explained:

[J]ust as the numerical value of a motion [*motus*] extending through time derives from an infinite number of impetuses, so, in turn, impetus itself (even though it is something momentary) arises from an infinite number of increments successively impressed on a given mobile thing. And so impetus too has a certain element from whose infinite repetition it can only arise.²²

This passage occurs in the lead-up to Leibniz's presentation of rotational and centrifugal force in the *Specimen* discussed above. Here, we encounter some different registers of the infinitesimal. Although Leibniz does not provide the mathematical specifics, he classifies these registers according to orders of differentiation according to the infinitesimal calculus: finite extended motion (in space and time) is composed by infinitesimal moments of impetus and (the quantitative value of) impetus is itself composed by solicitations corresponding to lower orders of differentials.²³ He explains, "[I]mpetus is the product of the bulk [*moles*] of a body and its velocity whose quantity is what the Cartesians usually call quantity of motion, that is, the momentary quantity of motion."²⁴ These impetuses integrate into a coherent continuous motion. He puts this in the following way, "[W]e can distinguish the present or instantaneous element of motion from the same motion extended through a period of time, and call the former *motio*."²⁵

²¹ In their editing of the *Specimen Dynamicum*, Ariew and Garber provide an earlier version of this passage where Leibniz employs the difference of order between a point and a line. Again, the issue is not how a line can be constituted by points but rather how this comparison provides an analogy for a difference in order or dimensionality; Leibniz, *Philosophical Essays*, fn. 121.

²² Leibniz, "*Specimen Dynamicum*," p. 121.

²³ The problem of how impetus and dead force are to be mathematically integrated to finite extended motion is controversial. Without intervening in the debate, I emphasize here only the difference of the order of differentials corresponding to their measures. See footnote 25 below. Cf. François Duchesneau, *La dynamique de Leibniz*, Paris: Vrin, 1994, p. 223 and Domenico Bertoloni Meli, *Equivalence and Priority*, Oxford: Oxford University Press, 2002, p. 90.

²⁴ Leibniz, "*Specimen Dynamicum*," p. 120.

²⁵ *Ibid.*

The terms above are articulated along a model provided by the infinitesimal calculus and can be arranged to show a series of integrations. In modern mathematical terms, if we were to take one moment of this impetus, it would be to take a first derivative of the motion at an instant, the momentaneous rate of change at an instant. A second register occurs at the level of impetuses themselves. As Leibniz explained, the state of an impetus at any given moment is an interaction of different forces that, due to the initial force of motion – the compounding solicitations of gravity and a number of other factors – modify the impetus itself. These different factors inflect motion in such a way that they determine how change (or motion *qua* change of place) itself changes. This observation of the changes in motion (a change of change), as well as the nature of *conatus*, corresponds to what we could associate with higher-order differentiations of finite extended motion. Outside of marking a difference between impetus and the changes in impetus, the mathematical details here should be reserved for a different context.²⁶ In Leibniz's terms, this second register of the infinitesimal concerns infinite repetition (like that of a "solicitation" to motion in the case of centrifugal force) which, in the case of a ball tied to a fixed center in a cylinder, remains constant and infinitely impresses this force

²⁶ Duchesneau ventures a mathematical expression of the relation between conatus, impetus, dead force and living force:

$$\text{Conatus} = dv = gdt$$

Pour l'*impetus* réduit à la quantité de mouvement dans l'instant (=quantité de motion), par contraste avec $m|v|$ pour Descartes:

$$\text{impetus} = \int_0^t gdt = mv$$

pour l'*impetus* dans son effet temporel:

$$\text{sommation temporelle d'impetus} = m \int_0^t gtdt = m \int_0^t vdt$$

pour la force morte:

$$\text{vis mortua} = m \int_0^t gdt = mv$$

pour la force vive:

$$\text{vis viva} = m \int_0^t gtdt = m \int_0^t vdt = ms = mv^2$$

For reasons given by Bertoloni Meli, I refrain from a direct mathematical interpretation of the relation between impetus and finite extended motion through the calculus of integration and differentiation although Leibniz is clearly drawing from the methods of the infinitesimal calculus. Despite this, I do have sympathy for François Duchesneau's mathematical formulation. For reasons that would be beyond the scope of this paper, it remains difficult to textually justify this as an exact expression of Leibniz's argument. Duchesneau, *La dynamique de Leibniz*, p. 223. Cf. Bertoloni Meli, *Equivalence and Priority*, pp. 89-92.

without actually causing any change in the velocity of the ball.²⁷ In view of the fact that Leibniz did invoke higher-order differentials, we have reason to suspect that Leibniz was indeed committed to a deep rather than superficial relation between the dynamical account of forces and the mathematical structure articulated by the infinitesimal calculus. Despite all this however, in Leibniz's conclusion to this paragraph, we find a familiar prudential remark on these mathematical quantities. "Nevertheless, I wouldn't want to claim on these grounds that these mathematical entities are really found in nature, but I only wish to advance them for making careful calculations through mental abstraction."²⁸ Daniel Garber notes that this very sentence was a late addition to the version of the *Specimen* eventually published in *Acta Eruditorum*.²⁹ One may speculate as to the reason behind this. What we can safely assume however is that Leibniz, in such a public context, wanted to display an unambiguous position concerning the existence of these infinitesimal terms *qua* mathematical entities. Of course this public position does not differ from his other remarks in private correspondence. As he quite regularly remarks, infinitesimal entities (even if we could place them in different registers) involved in the measure of the interaction between forces and motions are advanced for calculation's sake and through mental abstraction. These entities do not exist in nature as actual things even if they are capable of expressing quantitative relations between things.

If these infinitesimal terms do not exist in nature, what good are they for examining nature? In what follows, we will turn to an interpretation of these same passages through the notion of representation that has been argued by D. Garber in his recent article "Dead Force, Infinitesimals and the Mathematicization of Nature".³⁰ But without yet entering into an examination of Garber's interpretation, we should have in mind the two concerns, already invoked above, that constitute reasons for being wary of an interpretation through the lens of "representation".

The first concern is the question of what is actually being represented through mathematics in Leibniz's account in the *Specimen*. At least since the *Discourse on Metaphysics*, nine years before the *Specimen*, Leibniz had clearly expressed his view on the ontological dependence of locomotive phenom-

²⁷ The expression of *impetus* integrated in time joins with that of *vis viva*, as the expression of *conatus* integrated in the form of *impetus* in the instant joined with that of *vis mortua*." François Duchesneau, "Rule of Continuity and Infinitesimals in Leibniz's Physics," in Ursula Goldenbaum and Douglas Jesseph (eds.), *Infinitesimal Differences*, Berlin and New York: Walter de Gruyter, 2008, pp. 247-248. See figure 1.

²⁸ Leibniz, "*Specimen Dynamicum*," p. 121.

²⁹ Daniel Garber, "Dead Force, Infinitesimals and the Mathematicization of Nature," in *Infinitesimal Differences*, p. 302.

³⁰ *Ibid.*, pp. 281-306.

enon on its cause *qua* force that grants motion its degree of actuality.³¹ Since mathematical objects and the phenomenal aspect of corporeal motion (size, shape and motion) are both in the realm of the ideal, no real link between the ideal and actual is established or at least not through mathematics. On the contrary if Leibniz understood mathematics as adequate to the description of an actual metaphysics of motion that goes beyond phenomenon, then the worry about confounding the ideal and the actual appears once again.

This leads us to a second concern. As Leibniz famously states in his 30 June 1687 letter to Arnould, “what is not truly one being is not truly one being either”.³² This ontological criterion of unity *qua* actuality would appear to be opposed to the suitability of infinitesimal terms for the treatment of actual things. We might assume that if the *dynamics* aims to supply a coherent account of corporeal motion, then the representation of these actual terms must be separated from the “fictions” that are the infinite and infinitesimal. With these two concerns in mind, we turn to Garber’s interpretation of the infinitesimal at work in the *dynamics* as a case of representation.

4. *The Limits of Representation between Ideal and Actual*

In treating the same passages in the *Specimen* as I have in the presentation of dead force, Garber begins by noting Leibniz’s highly suggestive use of infinitesimal quantities. He remarks:

As Leibniz presents it in the *Specimen*, dead force would appear to be a real instantiation of an infinitesimal quantity, an infinitesimal magnitude that really exists in nature. But, of course, Leibniz is not inclined to take a realistic view of infinitesimal magnitudes. Is the reality of dead force consistent with the very skeptical attitude that he takes to the reality of infinitesimal magnitudes?³³

Garber continues on to note Leibniz’s aim to build a mathematical physics, a *dynamics*, and thus subjects these physical magnitudes to mathematics.³⁴ There is no doubt that this is correct but we are not dealing with just any mathematical term here but infinitesimal fictions. As such Garber responds to how infinitesimals are to operate in this context by referring us back to the basic distinction between a level of reality or actuality and an ideal level of mathematical description or representation. Garber makes it clear that he is relying on this distinction between the ideal and actual to make his point

³¹ See in particular 12th and 18th propositions of *Discourse on Metaphysics*.

³² GP II, p. 96.

³³ Garber, “Dead Force, Infinitesimals and the Mathematicization of Nature,” p. 280.

³⁴ *Ibid.*, p. 289.

about representation by remarking, through one of Leibniz's intended correspondence with Foucher that,

Leibniz draws a clear distinction between the world of mathematical entities (lines, surfaces, numbers), and the world of concrete things. The problem of the composition of the continuum is concerned with the parts from which continua can be constructed. Leibniz's point is that the mathematical continuum does not have such parts, nor does it need them: its parts come from the division of the line, and these parts are not properly elements of that line. However, in real *concreta*, the whole is indeed composed of parts, though those parts don't make up a genuine mathematical continuum. The problem of the composition of the continuum is thus solved: the objects of geometry, which exist in the realm of the ideal, are continuous, but not composed of parts; the real objects that exist in the physical world are composed of parts, but they are not continuous.³⁵

Here Leibniz's remarks to Foucher³⁶ correspond closely to those that we have considered above in his correspondence to De Volder. As such, Garber's interpretation of the use of infinitesimals explicitly relies on making use of the distinction between the ideal and actual in order to account for the use of infinitesimals on the features of corporeal motion. The important point that Garber adds to this distinction between mathematical entities and concrete reality is that the realm of reality and that of geometry cannot be absolutely distinct as if one has nothing to do with the other. Garber formulates this by saying, "Geometry in this way can be said to represent something that is really in body, even if it has properties that the concrete body it represents does not, such as continuity: mathematical representation is not identity."³⁷

To defend Leibniz's use of mathematics by underlining that the mathematical representation of actual things is not identical with these actual things is of course correct but this does not help identify why such a representation can be legitimate given the problem of the confounding of ideal and actual. Here, Garber makes use of the distinction between the ideal character of the continuous and the actual character of the discrete. Now, I have stressed in the above how this distinction is itself an intra-mathematical one. There are both discrete and continuous things in the ideal but only discrete ones in the actual. Garber underlines the fact that real objects are not continuous but leaves open the relation between continuous ideal things and non-continuous actual things. Of course this is not the same as suggesting that a direct correlation between ideal and actual should be made to the continuous and discrete. We would be in danger of this only if the ideal was exclusively continuous which is

³⁵ Ibid., p. 294.

³⁶ Leibniz, *Note on Foucher's Objection*, September 1695, GP IV, pp. 491-492; cited from Leibniz, *Philosophical Essays*, pp. 145-147.

³⁷ Garber, "Dead Force, Infinitesimals and the Mathematicization of Nature," p. 295.

far from the case. Instead, the problem with Garber's representational reading of this relation is that he interprets the distinction between the actual and the ideal too closely with the distinction between real and mathematical things. Under this framework, Leibniz's warning against confounding the ideal and the actual becomes construed as a warning against confounding mathematical terms and actual things. Although it is clear that one should not confound mathematical terms and actual things, this is not quite Leibniz's point here. Leibniz's point is rather that we should not confound ideal and actual things by searching for indeterminate continuous parts in the actual. Sticking too closely to Garber's distinction that places mathematical terms on one side of representation and actual things on the other side as the represented does not shed much light on what follows in Leibniz's letter to De Volder cited above.

For there to be a mathematical physics at all, as Garber rightly points out, some form of representation must be taking place between actual things and ideal mathematical terms. Yet representation by itself is too vague to grasp the relation that Leibniz aims to establish between mathematics and things, the ideal and actual, continuity and discreteness. Garber is right in asserting a separation between mathematics and actual things but the real question concerns what kind of separation is actually involved. Without understanding the nature of this separation, their relation, expressed vaguely until now as representation, will remain opaque. We thus turn to an examination of what Leibniz intended to "represent" in his mathematical account of corporeal motion.

5. *What is Being Represented?*

Given our quandaries above and critique of Garber's interpretation, I suggested that we must first understand the nature of the separation between mathematical terms and actual things before we can understand how they are related. This demands some clarification of what exactly it is that is being represented in the *Specimen* by mathematical terms.

What exactly is being represented by mathematical terms matters a great deal in understanding not only the passages in the *Specimen* but the background of this *dynamics* project as a whole. Here Garber himself turns our attention to Leibniz's critique of Cartesian mechanics that were already operative in Leibniz's thinking ten years before the *Specimen* in the *Discourse on Metaphysics*. More importantly, this criticism is also a form of self-criticism against Leibniz's own earlier positions on the nature of corporeal motion.³⁸ We can

³⁸ This claim is not to equate Leibniz's earlier position with a Cartesian one. Leibniz's earlier (pre 1672) reflections on bodies and their motions come from a Hobbesian inspiration. What Descartes and Hobbes share however is a science of motion based on geometrical relations: size, shape and extension. My emphasis here is on Leibniz's rejection of this approach to treating corporeal motion in general and not merely one or the other author.

begin by looking at how Garber employs representation in his interpretation of the difference between Cartesian mechanics and Leibniz's critique of it. Here Garber notes that the distinction made between mathematical terms and actual things is "exactly where Descartes erred, in confusing the mathematical representation of bodies in geometrical terms with their concrete reality."³⁹

By distinguishing Leibniz's approach from Descartes, Garber explains the mathematical sense of Leibniz's critique of the Cartesian position as well as the development of the concept of force by showing the real distinction between the reality of force and its mathematical formalization. Leibniz's turn to force positively designates a reality inherent to bodies, a reality that allowed him to disentangle force, as we see in the quote below, from the ambiguities of the appearance of extension in bodies and motion. This dimension also gives the relativity of motion a non-relative foundation in the immanence of forces in bodies. Here Garber quotes proposition 18 of the *Discourse on Metaphysics*:

For if we consider only what motion contains precisely and formally, that is, change of place, motion is not something entirely real, and when several bodies change position among themselves, it is not possible to determine, merely from a consideration of these changes, to which body we should attribute motion and rest [...] But the force or proximate cause of these changes is something more real, and there is a sufficient basis to attribute it to one body more than to another. Also, it is only in this way that we know to which body the motion belongs.⁴⁰

This anchoring of the relativistic interchange of bodies in motion to a metaphysical reality of force not only grants Leibniz access to a level of actual reality, that of force, but equally important, a way of understanding phenomenal or geometrical aspects of motion as less than real or actual. This reality of force is a basis for motion insofar as it anchors phenomenon to something more real and a determination more concrete than motion itself. Leibniz's fundamental aim in the *Specimen* was to explicate the properties of motion, which are ultimately relative, as being analyzable along the lines of this deeper reality of force and not vice-versa. It follows then that the mathematical dimension of the development of force in the *Specimen* was the attempt to measure the nature of this force with a mathematical apparatus, that is, to connect the profound reality of force with the (ideal) mathematical terms through which we generally treat motion and mass.

This feature of the *Specimen* thus helps clarify what it is that Leibniz aimed to represent through mathematical terms in this text. At the same time, however, it serves to complicate any simple distinction between the actual and ideal. Here Leibniz develops a distinction between two levels of what is

³⁹ Garber, "Dead Force, Infinitesimals and the Mathematicization of Nature," p. 295.

⁴⁰ Ibid., p. 51.

represented or capable of being represented by mathematical terms. On one level, we find the series of phenomena with various features of size, shape and magnitude that are immediately capable of being treated in mathematically. On another level, Leibniz's main aim of the *Specimen* was to understand the causes of these phenomena through an infra-phenomenal series of forces that allow us access to a non-relativistic and, according to Leibniz above, "more real" level of actual things. The problem here is that while we can grant the use of mathematics to treat the ideal or phenomenal aspects of motion by immediate mathematical representations of size, shape and extension, it is not clear why something infra-phenomenal like force should be susceptible to the same sort of analysis.

The separation of at least two levels in Leibniz's dynamical account of corporeal motion gives us reason again to emphasize the limits of Garber's interpretation. Again here, we find Garber's distinction between Leibniz's and Descartes' approach insufficient. It is true that since Descartes identified extended substances with its "principal attribute" of extension, geometry is the direct study of "extendedness" itself.⁴¹ As such, Descartes asserts confidently in the *Principles of Philosophy* that geometry and pure mathematics is all that is needed to study and understand nature.⁴² It would, however, be a mistake to understand Descartes as confusing mathematics and the thing represented mathematically. Employing his distinction between a real, modal and conceptual distinction, Descartes in the 8th principle of the 2nd part of *Principles* notes that,

We can, for example, consider the entire nature of the corporeal substance which occupies a space of ten feet without attending to the specific measurement.... And, conversely, we can think of the number ten, or the continuous quantity *ten feet*, without attending to this determinate substance.⁴³

This abstraction of mathematical or geometrical properties from extended and corporeal substance is a conceptual distinction not a real one. But of course this is exactly the kind of distinction that is being made in the very practice of arithmetic and geometry. Descartes held the unique and complete adequacy of geometry to grasp corporeal substance, that is, the entire realm of extended things. He did not however insist on the strict identity of extended things with the idea of extension or quantity. Further, when it comes to Descartes' own treatment of inertial forces, laid out in his three laws of nature in the second part of *Principles*, we find a metaphysical rather than geometrical account that relies to God's conservation of the created world. This reaffirms

⁴¹ Rene Descartes, "Principles of Philosophy," in *The Philosophical Writings of Rene Descartes* Vol. I, trans. by John Cottingham, Robert Stoothoff, Dugald Murdoch, Cambridge: Cambridge University Press, 1985, PI 53, p. 210.

⁴² Descartes, "Principles of Philosophy," PII 64, p. 247.

⁴³ Ibid., PII 8, p. 226.

the point that even if Cartesian mechanics is exclusively dependent on size, shape and motion, these geometrical terms merely *represent*, once again, something in substance that have metaphysical (divine) causes.⁴⁴

It is hence safe to say, with all the qualifications above, that Descartes saw doing mathematics (or geometry) as an activity that *represents* extended or corporeal substances. There is then no good reason to distinguish Leibniz and Descartes along the terms of "representation". Both of them conceived the use of mathematics as representative in a general sense. Of course, what it is *exactly* that mathematics represents turn out to be vastly different. Yet, this is another reason to consider the notion of "representation" as too vague an interpretive tool.

Within the limits of Leibniz's criticisms of Cartesian mechanics, Garber is right to underline that what Leibniz saw as a problem with Descartes' close association between geometrical extension and extended things. But in thinking through the difference between Descartes and Leibniz, we see how weak this distinction between ideal and the actual really is for satisfying the crucial distinction Leibniz aims to make in the *Specimen* between the phenomenon of motion, its underlying forces and how mathematical terms are to represent them. As we saw above, Garber's suggestion in contrasting Leibniz to Descartes was based on the notion that Descartes confused ideal and actual things while Leibniz did not. Now, even if we grant the idea that Descartes too closely associated the extended geometrical figures with actual extended things, the position advanced by Leibniz cannot be understood by merely focusing on this point. Leibniz's argument against the Cartesians is not so much that mathematical terms were mistaken to be actual things but rather that they reduced the account of (actual) corporeal things to a mathematical account through size shape and motion. Leibniz's counter-claim was that, "not everything conceived in body consists solely in extension and in its modification".⁴⁵ Hence the difference between Leibniz and Descartes on this question does not rest on their views on the usefulness of figures and geometry. Indeed Leibniz agrees with Descartes in part that, "all particular phenomena of nature can be explained mathematically or mechanically by those who understand them [...]"⁴⁶ Hence, the difference between Leibniz and Descartes is the rejection of a reductive account of corporeal motion to an account of the phenomenal or geometrical aspects of motion. Much of the motivation behind the *dynamics* project as a whole was his divergence from Descartes on this rejection of the primitiveness of extensionality for bodies and whether or not it should be treated more primitively by the notion of "force".⁴⁷ For Leibniz, a real account of motion is an account of their dynamic causes.

⁴⁴ Ibid., PII 37-40, pp. 240-242.

⁴⁵ Leibniz, "Discourse on Metaphysics," in *Philosophical Essays*, p. 51.

⁴⁶ Ibid.

⁴⁷ Cf. Leibniz, "Critical Thoughts on the General Part of the Principles of Descartes," GP IV, p. 364; cited from G.W. Leibniz, *Philosophical Papers and Letters*, 2nd Ed., ed. and trans. by

Through the terms above, we can understand that the difference between Cartesian mechanics and Leibnizian *dynamics* is not found in the conflation or distinction between mathematical terms and actual things but *how* they are distinguished. The difference is that Leibniz aimed at using his mathematical apparatus to penetrate a deeper, causal level of motion that he criticized the Cartesians for missing. As such, the distinction between the Cartesian and the Leibnizian use of mathematics is a disagreement over the *object* of the science of corporeal motion. In the *Specimen*, Leibniz sought to treat an infra-phenomenal level of reality that the Cartesians did not submit to mathematical analysis. As such, the distinction between the ideal and actual *qua* the distinction between mathematical terms and actual things does not clarify much. What our investigation demonstrates is rather that this distinction between ideal mathematical entities and actual corporeal things apply equally across Cartesian and Leibnizian conceptions. What actually distinguishes a Cartesian and Leibnizian approach is the complexity within their account of corporeal motion, a complexity that distinguishes what each thinker sought to represent through mathematical terms.

According to a loose idea of “representation” it is hard to show anyone as actually “confounding” the ideal and the actual. As such when Leibniz spoke of the confounding of ideal and actual as the source of the confusion that blinds one to the correct path in the labyrinth of the continuum, we cannot simply assume that any form of distinguishing the actual and the ideal will do. The simple understanding of the ideal as “not actual” does not directly lead us out of the labyrinth. It is in the critique of the Cartesians and his own earlier Hobbesian view that Leibniz sheds light on the sort of relation that Leibniz wished to establish between mathematics and the account of corporeal motion. Halfway through the first part of the *Specimen*, Leibniz described his earlier mistakes in thinking about the nature of body and motion. Remarking on his early work such as the 1671 *Theoria Motus Abstracti*, he admits that he had not yet recognized the idea that a stationary body resists a moving body that strikes it. His youthful (Hobbesian) position was that impact consists in the transferring of “*conatus*” of the moving body into the stationary body, causing the latter to move. The absurd result of this position was:

I showed that it ought to follow that the *conatus* of a body entering into a collision, however small it might be would be impressed on the whole receiving body, however large it might be, and thus, that the largest body at rest would be carried off by a colliding body however small it might be, without retarding it at all, since such a notion of matter contains not resistance to motion but indifference.⁴⁸

This same mistake also implies the absurd possibility of a perpetual motion machine.⁴⁹ Leibniz makes clear that a merely phenomenal treatment of motion is lacking precisely insofar as it is blind to the inherent force of resistance in bodies. A body of appropriately small mass repelled after striking a body of sufficiently large mass should leave the latter unmoved. It is this notion of an inherent nature of bodies that is one of the important points of departure that characterizes the *dynamics* project as such. The main consequence of this revised position, passing through the additional criticism of Cartesianism, was crystallized in the concept of force.⁵⁰ As such in the *Specimen*, Leibniz described his maturation from the earlier position:

Therefore, I concluded from this that, because we cannot derive all truths concerning corporeal things from logical and geometrical axioms alone, that is, from large and small, whole and part, shape and position, and because we must admit something metaphysical, something perceptible by the mind alone over and above that which is purely mathematical and subject to the imagination, and we must add to material mass [massa] a certain superior and, so to speak, formal principle. Whether we call this principle form or entelechy or force does not matter, as long as we remember that it can only be explained through the notion of forces.⁵¹

Leibniz asserts here that he does not care what it is called: form, entelechy, force. Leibniz is committed to something in bodies and their motion that is not “purely mathematical” and this distinguishes him from the Cartesian or Hobbesian idea of the adequacy of the mathematical description of nature. What Leibniz saw in his “addition” of a concept of material mass [massa] with its passive force of resistance is a new layer of forces that would, starting from the attention to the resistance or passive force of bodies, expand toward a systematic taxonomy of different forces. Leibniz’s critique here of his own earlier Hobbesian view is part and parcel of his general critique of Cartesian principles. Here Leibniz underlines that this metaphysical insight is what renders the project of a *dynamics* possible; this “formal principle” that is force can only be perceived intellectually rather than through imagination.

⁴⁹ Leibniz, “Preliminary Specimen: On the Law of Nature Related to the Power of Bodies,” GM VI, pp. 287–292; cited from Leibniz, *Philosophical Essays*, p. 108.

⁵⁰ To be clear, what Leibniz is referring to here, inertia or the passive force of resistance, should not be confused with the example of centrifugal force, an active force, treated above. In turn, neither of these problems should be confused with the problem of the structure of the collision of bodies to be treated below. A number of different forms of force (primitive, derivative, active, passive, total, partial, etc.) are systematically organized in the *Specimen Dynamicum*. Here my discussion is of force in general and hence I have not emphasized these different forces. I focus here only on the foundational distinction in the *Specimen* between force and the phenomenon of corporeal motion.

⁵¹ Leibniz, “*Specimen Dynamicum*,” p. 125.

This intellectual perception thus reveals something that in turn allows us to treat these mathematical terms more appropriately and understand phenomena more coherently.

Although we can understand Leibniz's *mathematical* maturation as the rejection of actual infinite and infinitesimal magnitudes, we should not view Leibniz's development of his account of corporeal motion as one that takes place along the distinction of the ideal and the actual. Rather, through the development of this "formal principle", Leibniz's *dynamics* aims to provide an account of what is actual in motion through its causes. This task requires the resources of ideal principles from metaphysics and ideal terms in mathematics. As such, what really changes in Leibniz cannot be adequately understood through the lens of the distinction between the ideal and actual. Rather Leibniz's maturation allowed him to conceive the role of mathematics in the problem of bodies and their motions as articulating the relations between force on the one hand, something actual and metaphysical which is intellectually perceived, and on the other hand, phenomenon, which is the perceptual outcome of these dynamical causes. As such, the distinction between the ideal and the actual is one that is, for Leibniz, to be found within his account of corporeal motion itself: the interplay between actual forces and their resulting imaginary phenomena. It is this interplay that is accounted for by the science of *dynamics*. In turn the role of mathematics here cannot be brushed over to the side of ideals but in fact allows Leibniz to articulate a relation between (ideal) phenomena and their (actual) causes.

6. *The Discrete and the Continuous in Representation*

If my analysis above is correct, we may permit ourselves to embed Leibniz's analogy of the actual and ideal with the discrete and continuous *within* the *dynamics* itself. In doing this, we demonstrate that the distinction between the actual and the ideal is one that operates within the dynamical account of corporeal motion such as to bridge the gap between the ideal and the actual within corporeal motion. This use of mathematics is ultimately a form of "representation" but my emphasis here is that Leibniz's use of mathematics here is not one that prudentially places ideal mathematical elements on one side of representation while actual physical elements belong to the side of the represented. More than mere "representation", the mathematical aspects of Leibniz account actively coordinate the ideal and actual aspects of Leibniz's account of corporeal motion, through force, in the *Specimen*.

To start, we recall that the distinction between actual and ideal can be made analogically along mathematical lines, the difference between the actual and ideal within the context of motion can thus be analyzed along the lines of the discrete and the continuous. Recalling his 1706 letter to Volder cited above, it should be emphasized that what is ideal accommodates both the

discrete and continuous but what is actual can only be unities and thus discrete. We can establish this extended but tentative analogy by seeing how the principle of continuity, following Leibniz's transformed notion of the inherent nature of bodies, is explained in the *Specimen*.

In the second section of the *Specimen*, a portion that was left unpublished during Leibniz's lifetime and unanalyzed by Garber in the article in question, Leibniz uses the principle of continuity to treat one of the central aims of the *Specimen*, the laws of impact. Here in the second part of the *Specimen*, Leibniz writes:

If one case continually approaches another case among the givens, and finally vanishes into it, then it is necessary that the outcomes of the cases continually approach one another in that which is sought and finally merge with one another.⁵²

This argument about continuity arrives at the moment when Leibniz aims to justify his claim that the laws governing motion and rest are to be understood consistently with one another. Recall from our brief discussion of the *Discourse on Metaphysics* and the *Specimen* above that this problem of motion and rest is precisely how Leibniz asserts the relativity of motion and motivates his treatment of the extensional features of motion as phenomenal and imaginary. With this in mind, we see how Leibniz argues here that the laws governing rest should be understood in terms of the limit of motion. In the case of the collision between two bodies, A and B, moving toward each other, the point of their collision would be understood as the momentaneous limit of motion, the point of rest for both bodies A and B. At the moment after this collision, the two bodies move apart, traveling away from each other after the mutual effect of forces in that moment of collision. Leibniz understood this mutual effect through the mutual deformation of their elastic bodies, and in turn, the restoring of their bodies from deformed shapes as they move away from one another. One of the main points that Leibniz wished to draw from this model of collision and elasticity was the following:

[I]t is already obvious how no change happens through a leap; rather, the forward motion diminishes little by little and after the body is finally reduced to rest, the backward motion at last arises [...]. And so, rest will not arise from motion, much less will motion in an opposite direction arise, unless body passes through all intermediate degrees of motion.⁵³

The idea that rest is a limit of motion was a position that Leibniz developed as early as 1671 in his letters to Oldenburg and to Arnauld during a period

⁵² Leibniz, "*Specimen Dynamicum*," p. 133.

⁵³ *Ibid.*, p. 132.

where, through Hobbesian inspiration, he understood the essence of bodies as motion.⁵⁴ This relationship between motion and rest played different roles in his arguments throughout the years. In earlier contexts, Leibniz held the idea that the rebounding of two bodies in collision could be explained by a reduction to an atomistic model where two absolutely hard bodies repel each other after having come to a full stop. In the case of the *Specimen*, a rejection of this earlier model was articulated along with the idea of a force inherent in bodies which allows for a continuous change (a change without leaps) in the context of collision. In this context, the eventual rebounding of the two bodies is preceded by their elastic deforming in a continuous way. Here Leibniz gives a new interpretation for how rest is nothing but the limit of motion:

Therefore, the case in which body A collides with the moving body B can be continuously varied so that, holding the motion of A fixed, the motion of B is assumed to be smaller and smaller, until it is assumed to vanish into rest, and then increase once again in the opposite direction.⁵⁵

With this new interpretation, Leibniz argues that rest is a limit of motion because motion is an effect that is relative to the underlying reality of the interplay of forces. With this basis in force, Leibniz renders rest as a limit of the continuous deformation of colliding bodies consistent with its contrary, motion. "By that very circumstance the motion itself is weakened, the force of the conatus having been transformed into their elasticity, until they are altogether at rest. Then, finally restoring themselves through their elasticity, they rebound from one another."⁵⁶ Here a continuous transformation of the position and shape of bodies, given to geometrical description, is systematized according to the conservation of the quantity of force. This allows us to bridge, through the notion of a limit, the gap between the *continuous* variations of phenomenon of motion and rest to a discrete actual cause. Phenomena would then vary relatively with respect to the motion and shape (the geometrical features) of the two bodies up to the limit of rest where there is no motion at all while force remains constant in the interchange. This is vastly superior to an atomistic account of the collision of hard bodies insofar as it allows two levels of exchange. There is first an exchange of forces which remains constant (within the system) and hence discrete throughout the exchange and there is secondly an elastic and continuous exchange where the rest engendered by the final moment of collision is a limit. The discrete interaction of forces is thus expressed as continuous transformations of shape and motion (velocity).

⁵⁴ Leibniz, Letter to Oldenburg, October 1671, A II, 1, N. 167; Letter to Arnould, November 1671, A II, 1, N. 173.

⁵⁵ Leibniz, "*Specimen Dynamicum*," p. 133.

⁵⁶ *Ibid.*, p. 132.

What this analysis reveals is that, with the anchoring of the dynamical account in force, the interplay of motion in phenomena and its underlying cause is opened up to features of both continuity and discreteness. That is, the continuity expressed by the phenomenon of colliding bodies, employing even the notion of rest as a limit, is something grounded by the actuality of force that underlie these transformations. In this, Leibniz's use of the methods of the infinitesimal calculus becomes highly relevant in his treatment of rest as the limit of motion through the continuous deformation of bodies in the context of impact. The role of infinitesimal terms in bridging the gap between imaginary phenomenon and the actuality of force is given a framework beyond what simple representation can explain. Although the fact that mathematical terms are not identical with actual things is more than clear, the nature of this representation however does not allow us to safely arrange mathematical terms on one side and actual things on the other. Here we see that it is mathematics that grants us the capacity to frame what is being separated between the ideal or continuous and the actual or discrete in motion but also how this separation can be made to relate. It is in these terms that we might finally understand what Leibniz means when he compares the negligible difference between infinitesimal terms and standard terms to the negligible difference between the actual and the ideal.

It is through these terms that we revisit the analogy with which this investigation began. The distinction of the actual and the ideal through the discrete and the continuous does not warrant a separation between the actual things and mathematical terms in a simple way. As we have pointed out, this distinction between ideal and actual is itself a mathematical one. In turn, in the *Specimen*, we see the role of the infinitesimal calculus as something that allows Leibniz to develop an account of both the difference and the relation between the various phenomena of motion and also the actual or causal reality through which they can be grounded.

What is actual in motion is force and what is ideal is found in the extensional features of bodies in motion: their positions, their velocity (direction and speed), their deformations in elastic collisions. The mathematics of infinitesimals thus enters into the picture precisely in the attempt to provide a theoretical framework where these ideal aspects of motion meet their actual and causal aspect: force. In our analysis, we see that mathematics *represents*, this is not in doubt. Yet, what is being represented, that is, the interplay between causes and phenomenal effects, is not one that can be simply treated as the representation of actual things by ideal entities, fictions. Rather these infinitesimal fictions provide a crucial bridge through which what is ideal and actual within corporeal motion can be brought together under the roof of a "new science". As such, Leibniz's application of the resources of his infinitesimal calculus provides a crucial aspect not only of how he distinguishes between the phenomenon of motion and its formal (or metaphysical) cause but

also how he brings them back together in the treatment of the (phenomenal) continuity of motion. According to the very distinction with which we began our investigation, what is actual can only be discrete unities and what is continuous in corporeal motion is a phenomenal effect engendered by force. In order to provide this conversion between force and motion, Leibniz's account requires a mathematical framework of the infinitesimal calculus including the notion of the limit, to guarantee this interchange between continuity to discreteness. Hence, rather than seeing the distinction between the actual and the ideal as what grounds infinitesimal terms, we see that it is a mathematics of the infinitesimal that grounds distinction between the actual and the ideal in the *dynamics*.

7. Concluding Remarks

What I have argued above is that the systematic relations between the finite and infinitesimal terms in the infinitesimal calculus provide the framework for Leibniz to coordinate a number of fundamental aspects of his dynamics: a classification of living, dead and other forces, an account of the relation between continuous motion and the discrete quantity of force preserved, and an explanation of rest as the limit of motion. Leibniz's use of the infinitesimal calculus provides a framework capable of organizing the general structure of the relations between an actual causal layer of forces and their ideal phenomenal effects.

If this analysis is correct, the notion of representation that requires us to correlate a series of representations and a series of represented things must be rejected in analyzing the role of mathematics in Leibniz's *Specimen*. The generality of the notion of representation here assures that the idea of mathematics representing actual things can, outside of extreme circumstances, never be flatly false. But, more importantly, this approach obscures the role of mathematics in Leibniz's work as well as renders unintelligible why the infinitesimal calculus is so operative in his mature *dynamics* project.

Employing a critique of Garber's recent article as *causa occasionalis*, I have pushed for a reading of Leibniz's use of infinitesimal terms in *Specimen* as one that places the ideal and the actual in a sort of convergence without confounding them. I have developed this through a critique of the notion of "representation" as the relation between actual and ideal. What emerges from this critique is the important role played by infinitesimals and the calculus as a means of coordinating what is ideal and actual within his account of corporeal motion.

This account is by no means complete. More work has to be done in order to fully address the role of Leibniz's use of mathematics in the *Specimen* and the larger *dynamics* project. My narrow focus on a concrete example of Leibniz's use of infinitesimals to treat actual things is meant to show that this

relation can be understood in a different way from the generally accepted one for separating ideal fictions from actual things. In this I have attempted to shed some light on how Leibniz brought his fictional infinitesimal to bear systematically on actual things and thus provide a model for further understanding the convergence between the ideal and the actual as conveyed in his 1706 letter to De Volder.

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